

NOTE ON A DETERMINANT WITH FACTORS LIKE THOSE OF THE DIFFERENCE-PRODUCT.

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1. The determinant in question, when of the 3rd and 4th order, is

$$\begin{vmatrix} 1 & a+b & ab \\ 1 & b+c & bc \\ 1 & c+d & cd \end{vmatrix}, \quad \begin{vmatrix} 1 & a+b+c & ab+ac+bc & abc \\ 1 & b+c+d & bc+bd+cd & bcd \\ 1 & c+d+e & cd+ce+de & cde \\ 1 & d+e+f & de+df+ef & def \end{vmatrix},$$

the scheme of construction being readily grasped when the scheme for the first row and the scheme for the second column have been observed. Clearly it is such that when of the n^{th} order the number of variables occurring in any row is $n-1$, and the total number of variables is $2(n-1)$.

2. Recalling the effect produced on a determinant by reversing the order of its rows, we see that the function under consideration, F say, is at most altered only in sign when the order of its variables is reversed : for example,

$$\begin{aligned} F(a \ b \ c \ d) &= -F(d \ c \ b \ a), \\ F(a \ b \ c \ d \ e \ f) &= F(f \ e \ d \ c \ b \ a). \end{aligned}$$

3. Performing on the 4-line determinant the operations

$$\text{row}_1 - \text{row}_2, \text{row}_2 - \text{row}_3, \text{row}_3 - \text{row}_4$$

we obtain

$$\begin{aligned} F(a \ b \ c \ d \ e \ f) &= - \begin{vmatrix} a-d & (b+c)(a-d) & bc(a-d) \\ b-e & (c+d)(b-e) & cd(b-e) \\ c-f & (d+e)(c-f) & de(c-f) \end{vmatrix} \\ &= -(a-d)(b-e)(c-f) \cdot F(b \ c \ d \ e), \\ &= (d-a)(e-b)(f-c) \cdot F(b \ c \ d \ e), \end{aligned}$$

where it will be observed that the three factors on the right are got by, as it were, subtracting

$$\begin{array}{r} \text{from the row } d \ e \ f \\ \text{the row } a \ b \ c, \end{array}$$

and the F on the right from that on the left by removing from the latter its first and last variables.

Exactly similarly it is found that

$$F(a \ b \ c \ d \ e \ f \ g \ h) = (e-a)(f-b)(g-c)(h-d) \cdot F(b \ c \ d \ e \ f \ g),$$

and we thus finally reach

$$\begin{aligned} F(a \ b \ c \ d \ e \ f \ g \ h) &= (e-a)(f-b)(g-c)(h-d) \\ &\quad \cdot (e-b)(f-c)(g-d) \\ &\quad \cdot (e-c)(f-d) \\ &\quad \cdot (e-d). \end{aligned}$$

4. The marked resemblance of this development to that of the alternant

$$| a^0 \ b^1 \ c^2 \ d^3 \ e^4 |$$

cannot but strike the student, the factors in both cases being all *differences*, and the number of them in the one case being the same as in the other. We are thus led to expect that the said alternant is derivable from $F(a, b, \dots, h)$ by specialization; and such is readily found and shown to be the case. For, returning to the development at the close of the preceding paragraph, and putting

$$f, g, h = a, b, c$$

we obtain

$$\begin{aligned} &(e-a)(a-b)(b-c)(c-d) \\ &\quad \cdot (e-b)(a-c)(b-d) \\ &\quad \cdot (e-c)(a-d) \\ &\quad \cdot (e-d). \end{aligned}$$

$$\text{i.e.} \quad (-1)^6 \cdot (b-a)(c-b)(c-a)(d-c)(d-b)(d-a)(e-d)(e-c)(e-b)(e-a),$$

so that our result is

$$F(a \ b \ c \ d \ e \ a \ b \ c) = | a^0 \ b^1 \ c^2 \ d^3 \ e^4 | = \zeta^{\frac{1}{2}}(a \ b \ c \ d \ e).$$

5. An interesting alternative mode of proving this last equality is to show that

$$F(a \ b \ c \ d \ e \ a \ b \ c) \cdot | a^0 \ b^1 \ c^2 \ d^3 \ e^4 | = | a^0 \ b^1 \ c^2 \ d^3 \ e^4 |^2.$$

Taking for shortness' sake determinants of the 4th order we have

$$\begin{aligned} &\begin{vmatrix} 1 & a+b+c & ab+ac+bc & abc \\ 1 & b+c+d & bc+bd+cd & bcd \\ 1 & c+d+a & cd+ca+da & cda \\ 1 & d+a+b & da+db+ab & dab \end{vmatrix} \cdot \begin{vmatrix} a^3 & -a^2 & a & -1 \\ b^3 & -b^2 & b & -1 \\ c^3 & -c^2 & c & -1 \\ d^3 & -d^2 & d & -1 \end{vmatrix} \\ &= \begin{vmatrix} \cdot & \cdot & \cdot & (d-a)(d-b)(d-c) \\ (a-b)(a-c)(a-d) & \cdot & \cdot & \cdot \\ \cdot & (b-c)(b-d)(b-a) & \cdot & \cdot \\ \cdot & \cdot & (c-d)(c-a)(c-b) & \cdot \end{vmatrix} \\ &= - | a^0 \ b^1 \ c^2 \ d^3 |^2; \end{aligned}$$

and consequently the multiplicand

$$= -[a^0 b^1 c^2 d^3].$$

It is also just worth noting that the one determinant can be directly transformed into the other, the first step of the process being to perform on $F(a\ b\ c\ d\ a\ b)$ the operation

$$\text{col}_2 - (a+b+c+d) \text{col}_1.$$

6. Besides the specialization

$$f, g, h = a, b, c$$

there are a number of others resembling it although of less interest: for, without causing the function to vanish, we may put

$$\begin{aligned} f &= a \text{ or } e, \\ g &= a \text{ or } b \text{ or } e, \\ h &= a \text{ or } b \text{ or } c \text{ or } e; \end{aligned}$$

for example,

$$F(a\ b\ c\ d\ e\ e\ e\ e) = (e-a)(e-b)^2(e-c)^3(e-d)^4.$$

7. Of far greater interest, however, is a specialization of a different kind, namely, that which leads to a new determinantal representation for the product of the binomial sums of $a, b, c, \dots$. It is well known that for representing such a product as

$$\begin{aligned} &(a+b)(a+c)(a+d) \\ &\quad \cdot (b+c)(b+d) \\ &\quad \cdot (c+d) \end{aligned}$$

there is no form analogous to the alternant, and that in fact when the said product does occur in determinantal analysis it appears as a quotient, namely,

$$[a^0 b^2 c^4 d^6] \div [a^0 b^1 c^2 d^3].$$

8. Going back to the result of § 3 and altering the signs of the second triad of variables, we obtain

$$\begin{aligned} F(a, b, c, -d, -e, -f) &= (-d - a)(-e - b)(-f - c) \\ &\quad \cdot (-d - b)(-e - c) \\ &\quad \cdot (-d - c) \\ &= (-1)^6 \cdot (d+a)(e+b)(f+c) \\ &\quad \cdot (d+b)(e+c) \\ &\quad \cdot (d+c); \end{aligned}$$

and similarly for all other orders of F . In other words, *By altering the*

signs of the second $n-1$ variables in $F(a, b, c, \dots)$ all the differences in the factorial development of F are changed into sums.

9. From this as an immediate consequence we have

$$\begin{aligned} F(a, b, c, -d, -a, -b) &= (d+a)(a+b)(b+c) \\ &\quad \cdot (d+b)(a+c) \\ &\quad \cdot (d+c) \\ &= \text{product of the binomial sums of} \\ &\quad a, b, c, d : \end{aligned}$$

and similarly for every other such product. We can thus formulate the theorem that *If in the new determinantal equivalent for the difference-product of n quantities we alter the signs of the second $n-1$ variables in the functional symbol, the resulting determinant is equal to the product of the binomial sums of the said n quantities*: for example, from the equality of § 4

$$F(a, b, c, d, e, a, b, c) = | a^0 b^1 c^2 d^3 e^4 |$$

we deduce the companion equality

$$F(a, b, c, d, -e, -a, -b, -c) = | a^0 b^2 c^4 d^6 e^8 | \div | a^0 b^1 c^2 d^3 e^4 |.$$

10. There is an entirely different way of viewing the product of binomial sums, namely, as the eliminant of a pair of equations in x . Taking, for example, the equations

$$\left. \begin{aligned} (a+b+c+d+e)x^4 + (abc + \dots + cde)x^2 + abcde &= 0 \\ x^4 + (ab + \dots + de)x^2 + (abcd + \dots + bcde) &= 0 \end{aligned} \right\}$$

where the coefficients are sums of combinations of a, b, c, d, e , it is not difficult to show that their eliminant is the 10-factor product

$$(a+b)(a+c) \dots (d+e) \dots \dots \dots (E)$$

One sure basis for proof lies in the fact that, if we multiply both sides of the second equation by x and perform addition we obtain

$$(x+a)(x+b)(x+c)(x+d)(x+e) = 0,$$

and that the substitution of any one of the five roots of this derived equation in the two original equations changes their left-hand members into the product of four of the factors of E . Another basis of proof is the fact that if in the original equations we put one of the ten factors equal to 0, say $a+b=0$, the equations are transformable into

$$\left. \begin{aligned} (x^2+ab) \left(x^2 + \frac{cde}{c+d+e} \right) &= 0 \\ (x^2+ab)(x^2+cd+ce+de) &= 0 \end{aligned} \right\}$$

and are thus seen to be consistent.

The latter mode is less readily generalizable than the former. To give an indication of the line it takes, it may be stated that in the case of 7 variables the original equations

$$\begin{aligned}(a + \dots + g)x^6 + (abc + \dots + efg)x^4 + (abcde + \dots + cdefg)x^2 + abcdefg &= 0 \\ x^6 + (ab + \dots + fg)x^4 + (abcd + \dots + defg)x^2 + (abcdef + \dots + bcdefg) &= 0\end{aligned}$$

are transformed into

$$\begin{aligned}(x^2 + ab)\{ (cdefg)_1 x^4 + (cdefg)_3 x^2 + (cdefg)_5 \} &= 0 \\ (x^2 + ab)\{ \quad \quad \quad x^4 + (cdefg)_2 x^2 + (cdefg)_4 \} &= 0\end{aligned}$$

where, be it remarked as a digression in passing, the trinomial factors in the left-hand members are exactly similar to the left-hand members of the pair of equations with which we started this paragraph.

11. As a natural consequence of the result here foreshadowed we are led to consider the dialytic eliminant

$$\begin{vmatrix} 5_1 & 5_3 & 5_5 & . \\ 1 & 5_2 & 5_4 & . \\ . & 5_1 & 5_3 & 5_5 \\ . & 1 & 5_2 & 5_4 \end{vmatrix}$$

where

$$\begin{aligned}5_1 &\text{ stands for } a+b+c+d+e, \\ 5_2 &\dots \quad ab+ac+\dots+de,\end{aligned}$$

and so on.

Multiplying it column-wise by unity in the form

$$\begin{vmatrix} 1 & . & . & . \\ -e & 1 & . & . \\ . & -e & 1 & . \\ . & . & -e & 1 \end{vmatrix}$$

and then multiplying row-wise the resulting product by unity in the form

$$\begin{vmatrix} 1 & . & . & . \\ e^2 & 1 & . & . \\ e^4 & e^2 & 1 & . \\ e^6 & e^4 & e^2 & 1 \end{vmatrix}$$

we reach the result

$$\begin{vmatrix} 4_1 & 4_3 & . & . \\ 1 & 4_2 & 4_4 & . \\ . & 4_1 & 4_3 & . \\ . & 1 & 4_2 & e^4 + 5_2 e^2 + 5_4 \end{vmatrix},$$

the elements taking the simple form here given by reason of the equality

$$5_m - e5_{m-1} + e^2 5_{m-2} - \dots = 4_m.$$

Since the last element

$$\begin{aligned} e^4 + 5_2 e^2 + 5_4 &= e^4 + (e^4_1 + 4_2) e^2 + (e^4_3 + 4_4) \\ &= e^4 + e^3 4_1 + e^2 4_2 + e^4_3 + 4_4 \\ &= (e+a)(e+b)(e+c)(e+d), \end{aligned}$$

and its cofactor is our dialytic eliminant of the next lower order, it is clear that we shall have finally

$$\begin{vmatrix} 5_1 & 5_3 & 5_5 & . \\ 1 & 5_2 & 5_4 & . \\ . & 5_1 & 5_3 & 5_5 \\ . & 1 & 5_2 & 5_4 \end{vmatrix} = \begin{aligned} &= (e+a)(e+b)(e+c)(e+d) \\ & . (d+a)(d+b)(d+c) \\ & . (c+a)(c+b) \\ & . (b+a), \end{aligned}$$

as was expected.

12. A less pleasing but more quickly effective mode of proof would be to attest the existence of any one of the linear factors, say $d+e$, by showing that the putting of $d+e$ equal to 0 in the determinant causes the latter to vanish. Or, again, we might perform the operation

$$\text{col}_4 - de \cdot \text{col}_3 + d^2 e^2 \cdot \text{col}_2 - d^3 e^3 \text{col}_1,$$

when we should find that the 4th column as thus altered contains the factor $d+e$.*

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* Apparently the first to observe this peculiar bigradient was Mr. A. M. Nesbitt. See Educ. Times, lvii (1904), p. 490.